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Symmetry classifications and reductions of some classes of $(2+1)$-nonlinear heat equation

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Abstract

The $(2+1)$-nonlinear heat equation $u_t - f(u)(u_{xx} + u_{yy}) = 0$ is considered. A symmetry classification of the equation using Lie group method is presented and reduction to the first- or second-order ordinary differential equations is provided.

Keywords: Two-dimensional nonlinear heat equation; Symmetry classification

1. Introduction

The one-dimensional heat equation is extensively studied from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation can be found in Cantwell [1], Ibragimov [2] and Bluman and Kumei [3]. Since thermal diffusivity of some materials may be a function of temperature, it introduces nonlinearities in the heat equation that models such phenomenon. This shows that whereas nonlinear heat equation models real world problems the best, it may be difficult to tackle such problems by usual methods. In an attempt to study nonlinear effects Saied and Hussain [4] gave some new similarity solutions of the $(1+1)$-nonlinear heat equation. Later Clarkson and Mansfield [5] studied classical and nonclassical symmetries of the $(1+1)$-heat equation and gave new reductions for the linear heat equation and a catalogue of closed-form solutions for a special choice of the function $f(u)$ that appears in their model. In higher dimensions Servo [6] gave some conditional symmetries for a nonlinear heat equation while Goard et al. [7] studied the nonlinear heat equation in the degenerate case. Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [8,9]. An account of some interesting cases is given by Polyanin [10].

As pointed out above, the thermal diffusivity of materials such as gases is not a constant, but depends upon the temperature of the body. Physically it is quite an interesting situation and can be modelled by $(2+1)$-nonlinear heat equation

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$$u_t - f(u)(u_{xx} + u_{yy}) = 0, \tag{1.1}$$

where $f(u)$ is an arbitrary function of the variable u. Although this equation does not adequately model temperature dependence of thermal diffusivity, it can serve as a starting point for such situations. It models situations where variations in the temperature and thermal diffusivity is relatively small so that the product terms such as $f_u u_x$ can be ignored. In this paper a symmetry classification of (1.1) is presented using the Lie group method. We show that the two-dimensional subalgebra of Lie point symmetry generators reduces the equation to first- or second-order ordinary differential equations. In some cases the reduced ordinary differential equation is linear whose solution can be obtained easily. In the case that the reduced ordinary differential equation is nonlinear, solutions may be obtained using standard methods such as 'perturbation' method or a further Lie symmetry analysis can be performed.

2. Derivation of symmetries

In order to derive symmetry generators of Eq. (1.1) and obtain closed-form solutions for all $f(u)$, we consider one parameter Lie point transformation that leaves (1.1) invariant. This transformation is given by [11]

$$\tilde{x}^i = x^i + \epsilon \xi^i(x, y, t, u) + O(\epsilon^2), \quad i = 1, \ldots, 4, \tag{2.1}$$

where $\xi^i = \frac{\partial \tilde{x}^i}{\partial \epsilon}|_{\epsilon=0}$ [3] defines the symmetry generator associated with (2.1) given by

$$V = \xi \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial x^i} + \tau \frac{\partial}{\partial x^i} + \phi \frac{\partial}{\partial x^i}. \tag{2.2}$$

In order to determine four components ξ^i, we prolong V to second order. This prolongation is given by the formula [3,11]

$$V^{(1)} = V + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yt} \frac{\partial}{\partial u_{yt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}. \tag{2.3}$$

In above expression every coefficient 'of the prolonged generator' is a functions of (x, y, t, u) and can be determined by the formulae [3],

$$\phi^i = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}, \tag{2.4}$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij}, \tag{2.5}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of (1.1) we now use symmetry criterion for partial differential equations [11]. For heat equation this criterion is expressed by the formula $V^{(1)}[u_t - f(u)(u_{xx} + u_{yy})] = 0$ whenever $u_t = f(u)(u_{xx} + u_{yy})$. Using this symmetry criterion with (2.3) in mind immediately yields

$$\phi^t - f_u(u_{xx} + u_{yy})\phi - f(u)(\phi^{xx} + \phi^{yy}) = 0. \tag{2.6}$$

At this stage we calculate expression for ϕ^t, ϕ^{xx} and ϕ^{yy} using (2.4)–(2.5), substitute them in (2.5) and then compare coefficients of various monomials in derivatives of 'u.' This yields the following system of over-determined partial differential equations:

$$\tau = \tau(t), \tag{2.7}$$

$$\eta_u = \xi_u = 0, \tag{2.8}$$

$$\eta_x = -\xi_y, \tag{2.9}$$

$$\phi = \alpha(x, y, t)u + \beta(x, y, t), \tag{2.10}$$

$$f(u)(2\phi_{ux} - \xi_{xx} - \xi_{yy}) + \xi_t = 0, \tag{2.11}$$

$$f(u)(2\phi_{uy} - \eta_{xx} - \eta_{yy}) + \eta_t = 0, \tag{2.12}$$

$$f_u \phi(x, y, t) = f(2\xi_x - \tau_t), \tag{2.13}$$

$$f_u \phi(x, y, t) = f(2\eta_y - \tau_t), \tag{2.14}$$

$$\phi_t - f(u)(\phi_{xx} + \phi_{yy}) = 0. \tag{2.15}$$

To find a complete solution of the above system we start from (2.13) after writing it in the form

$$\phi = \frac{f}{f_u}(2\,\xi_x - \tau_t) \tag{2.16}$$

and then considering all possible cases in $f_u \neq 0$. Note that the case $f(u) = \text{constant}$ is of no interest because this choice reduces the heat equation to a linear one.

3. Classification of symmetries

In this section we give a classification of symmetries of the nonlinear heat equation (1.1). For this we begin by considering (2.16), which yields following *two* cases:

$$\text{I.}\quad \frac{f}{f_u} = A, \tag{3.1}$$

$$\text{II.}\quad \frac{f}{f_u} = g(u), \tag{3.2}$$

where A is a constant. We consider these possibilities separately.

3.1. Case I

To determine $f(u)$ in this case we integrate (3.1) with respect to 'u' to obtain

$$f(u) = Ke^{Au}, \tag{3.3}$$

where 'K' is an integration constant. Now differentiating (2.16) with respect to 'u' and using the resulting expression into (2.10) gives

$$\phi = \beta = A(\xi_x - \tau_t). \tag{3.4}$$

By substituting above expressions in (2.12)–(2.13) we obtain $\xi_t = 0 = \eta_t$. After some more manipulations one finds that ξ and η become

$$\begin{aligned}
\xi &= c_0 + c_1 x + c_2 y + 2c_3 xy + c_4\left(x^2 - y^2\right),\\
\eta &= c_5 - c_2 x + c_1 y + 2c_4 xy - c_3\left(x^2 - y^2\right).
\end{aligned} \tag{3.5}$$

The remaining equations can then be used to determine τ and ϕ as

$$\begin{aligned}
\tau &= c_6 t + c_7,\\
\phi &= 2Ac_1 - Ac_6 + 4Ac_4 x + 4Ac_3 y.
\end{aligned} \tag{3.6}$$

At this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators given by

$$\begin{aligned}
&V_0 = \frac{\partial}{\partial x}, \qquad V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2A\frac{\partial}{\partial u}, \qquad V_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
&V_3 = 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2\right)\frac{\partial}{\partial y} + 4Ay\frac{\partial}{\partial u}, \qquad V_4 = \left(x^2 - y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 4Ax\frac{\partial}{\partial u},\\
&V_5 = \frac{\partial}{\partial y}, \qquad V_6 = t\frac{p}{\partial t} - A\frac{\partial}{\partial u}, \qquad V_7 = \frac{\partial}{\partial t}.
\end{aligned} \tag{3.7}$$

It is easy to check that the symmetry generators found in (3.7) form a closed Lie algebra whose commutation relations are given in Table 1.

We now briefly show steps involved in the reduction of the nonlinear heat equation to a second-order differential equation. Since reduction under all the subalgebras cannot be given in the paper, we restrict ourselves to giving reductions in two cases only, i.e., $\{V_0, V_6\}$ and $\{V_1, V_3\}$. Reduction in the remaining cases is listed in the form of Appendices A and B at the end of the paper.

Table 1
Commutation relations satisfied by generators in case I

$[V_i, V_j]$	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_0	0	V_0	$-V_5$	$2V_2$	$2V_1$	0	0	0
V_1	$-V_0$	0	0	V_3	V_4	$-V_5$	0	0
V_2	V_5	0	0	$-V_4$	V_3	V_0	0	0
V_3	$-2V_2$	$-V_3$	V_4	0	0	$-2V_1$	0	0
V_4	$-2V_1$	$-V_4$	$-V_3$	0	0	$2V_2$	0	0
V_5	0	V_5	$-V_0$	$2V_1$	$-2V_2$	0	0	0
V_6	0	0	0	0	0	0	0	$-V_7$
V_7	0	0	0	0	0	0	V_7	0

3.1.1. Reduction under V_0 and V_6

From Table 1 we find that the given generators commute $[V_0, V_6] = 0$. Thus either of V_0 or V_6 can be used to start the reduction with. For our purpose we begin reduction with V_0. The characteristic equation associated with this generator is $\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$. Following standard procedure we integrate the characteristic equation to get three similarity variables

$$s = y, \qquad r = t, \qquad w(r, s) = u. \tag{3.8}$$

Using these similarity variables Eq. (1.1) can be recast in the form

$$w_r = K e^{Aw} w_{ss}. \tag{3.9}$$

At this stage we express V_6 in terms of the similarity variables defined in (3.8). It is straightforward to note that V_6 in the new variables takes the form

$$\widetilde{V}_6 = r\frac{\partial}{\partial r} - A\frac{\partial}{\partial w}. \tag{3.10}$$

The characteristic equation for $\widetilde{V}_6$ is $\frac{dr}{r} = \frac{ds}{0} = \frac{dw}{-A}$. Integrating this equation as before leads to new variables $\alpha = s$ and $\beta(\alpha) = r^A e^w$, which reduce (3.9) to a second-order differential equation

$$\beta^2 A = K\beta^A\big(\beta'^2 - \beta\beta''\big). \tag{3.11}$$

3.1.2. Reduction under V_1 and V_3

In this case the two symmetry generators V_1 and V_3 satisfy the commutation relation $[V_1, V_3] = V_3$. This suggests that reduction in this case should start with V_3. The similarity variables are $s = \frac{x^2+y^2}{x}$, $r = t$ and $u = \ln(x^{2A} w)$ and K is a constant. The corresponding reduced partial differential equation is

$$w_r = K w^A\left\{-2A + s^2\left(\frac{w_{ss}}{w} - \frac{w_s^2}{w^2}\right) + 2s\frac{w_S}{w}\right\}. \tag{3.12}$$

The transformed V_1 is

$$\widetilde{V}_1 = s\frac{\partial}{\partial s}. \tag{3.13}$$

The invariants of $\widetilde{V}_1$ are $\alpha = r$ and $\beta(\alpha) = w$ which reduce (3.12) to the ordinary differential equation

$$\beta' = -2AK\beta^A. \tag{3.14}$$

Reductions in remaining cases using generators forming subalgebra are given in the form of Table 3 in Appendix A.

3.2. Case II

In this case we classify solutions of the heat equation by considering (3.2), viz.,

$$\left(\frac{f}{f_u}\right)_u \neq 0.$$

Table 2
Commutation relations satisfied by generators in case II

$[V_i, V_j]$	V_0	V_1	V_2	V_3	V_4	V_5
V_0	0	V_3	V_0	0	0	0
V_1	$-V_3$	0	0	V_0	0	0
V_2	$-V_0$	0	0	$-V_3$	0	0
V_3	0	$-V_0$	V_3	0	0	0
V_4	0	0	0	0	0	$-V_5$
V_5	0	0	0	0	V_5	0

For this purpose we consider Eqs. (2.7)–(2.15). Following the procedure adopted in case I, it is easy to find that the components ξ, η, τ and ϕ of infinitesimal symmetry generator V become

$$\xi = c_0 - c_1 y + c_2 x, \qquad \eta = c_3 + c_1 x + c_2 y, \qquad \tau = c_4 t + c_5, \qquad \phi = u(2c_2 - c_4).$$

From above the following six symmetry generators can be constructed,

$$V_0 = \frac{\partial}{\partial x}, \qquad V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}, \qquad V_3 = \frac{\partial}{\partial y},$$
$$V_4 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \qquad V_5 = \frac{\partial}{\partial t}.$$

The commutation relations satisfied by the above generators are listed in Table 2.

Each of the two-dimensional subalgebra reduces the partial differential equation (1.1) to an ordinary one. We give a complete table of reductions to ordinary differential equations for all subalgebras in Appendix B. Reduction to ordinary differential equations in two cases is given below. As in previous case we show reductions under one commuting and one non-commuting sub-algebra.

3.2.1. Reduction under subalgebra $[V_1,\ V_5] = 0$ and $[V_3, V_2] = V_3$

Consider the algebra given by V_1 and V_5. Since $[V_1, V_5] = 0$, we begin reduction with $V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. The similarity variables for this generator are given by $s = x^2 + y^2$, $r = t$ and $w(r,s) = u$. Using these variables (1.1) reduces to a partial differential equation with two independent and one dependent variable given by

$$w_r = 4f(w)(sw_{ss} + w_s). \tag{3.15}$$

In order to perform second reduction of the above equation, we firstly write V_5 in terms of new variables $r,\ s$ and $w(r,s)$ to get $\widetilde{V}_5 = \frac{\partial}{\partial r}$. Corresponding to the above generator the similarity variables become $\alpha = s$ and $w(r,s) = \beta(\alpha)$, which reduce (3.15) to $\alpha\beta'' + \beta' = 0$. Similarly a reduction can be obtained using the subalgebra $[V_3, V_2] = V_3$. Here we need to start with $V_3 = \frac{\partial}{\partial y}$. That reduces (1.1) to the partial differential equation

$$w_{rr} + w_{ss} = 0. \tag{3.16}$$

Now transforming V_2 in these new variables for the second reduction we get

$$\widetilde{V}_2 = s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r} + 2w\frac{\partial}{\partial w}. \tag{3.17}$$

The invariants of $\widetilde{V}_2$ are $\alpha = \frac{r}{s}$ and $w = s^2\beta(\alpha)$. In these variables Eq. (3.16) reduces to the linear second-order differential equation given by

$$\left(1 + \alpha^2\beta''\right) - 2\alpha\beta' + 2\beta = 0.$$

4. Conclusion

The details and method presented here set the scene for further interesting studies regarding nonlinear and other partial differential equations arising from problems in mathematical physics. In particular the nonlinear heat equation containing a source term could also be considered.

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Appendix A

Table 3

Algebra	Reduction
$[V_0, V_1] = V_0$	$\beta' = 0$
$[V_0, V_6] = 0$	$K\beta^{\widetilde{A}}(\beta\beta'' - \beta'^2) + \beta^2 A = 0$
$[V_0, V_7] = 0$	$\beta'' = 0$
$[V_1, V_2] = 0$	$\beta' = 0$
$[V_1, V_3] = V_3$	$\beta' + 2AK\beta^{\widetilde{A}} = 0$
$[V_1, V_4] = V_4$	$\beta' + AK\beta^{\widetilde{A}} = 0$
$[V_1, V_5] = -V_5$	$\beta' = 0$
$[V_1, V_6] = 0$	$K\beta^{\widetilde{A}}\{(1-\alpha^2)\beta\beta'' - 4A\alpha\beta\beta' - 2A\beta^2 - (1+\alpha^2)\beta'^2\} + A\beta^2 = 0$
$[V_1, V_7] = 0$	$\alpha^2\beta'' + 2\alpha\beta' = \frac{2A}{\alpha^2} - \frac{\beta''}{\beta} + \frac{\beta'^2}{\beta^2}$
$[V_2, V_6] = 0$	$\beta''(\alpha^2+1) + 2\beta'\alpha - \frac{2A}{\alpha^2} = 0$
$[V_2, V_7] = 0$	$\alpha\beta'' + \beta' = 0$
$[V_3, V_4] = 0$	$\beta' = 0$, $A = 1$
$[V_3, V_6] = 0$	$\alpha\beta\beta'' + 2\beta\beta' - \alpha\beta'^2 = 0$, $A = 0$
$[V_3, V_7] = 0$	$\alpha^2\beta\beta'' + 2\alpha\beta\beta' - \alpha^2\beta'^2 - 2A\beta^2 = 0$
$[V_4, V_6] = 0$	$\alpha^2\beta\beta'' + 2\alpha\beta\beta' - \alpha^2\beta'^2 = 0$, $A = 0$
$[V_4, V_7] = 0$	$\alpha^2\beta\beta'' + 2\alpha\beta\beta' - \alpha^2\beta'^2 - A\beta^2 = 0$
$[V_5, V_6] = 0$	$K\beta^{\widetilde{A}}(\beta\beta'' - \beta'^2) + A\beta^2 = 0$
$[V_5, V_7] = 0$	$\beta'' = 0$

Appendix B

Table 4

Algebra	Reduction
$[V_0, V_2] = V_0$	$\beta' - 2\beta^2 = 0$
$[V_0, V_3] = 0$	$\beta' = 0$
$[V_0, V_4] = 0$	$\beta'' = 0$
$[V_0, V_5] = 0$	$\beta'' = 0$
$[V_1, V_2] = 0$	$\beta' - 4\beta^2 = 0$
$[V_1, V_4] = 0$	$\alpha\beta'' + \beta' = -\frac{1}{4}$
$[V_1, V_5] = 0$	$\alpha\beta'' + \beta' = 0$
$[V_3, V_2] = V_3$	$(1+\alpha^2)\beta'' - 2\alpha\beta' + 2\beta = 0$
$[V_2, V_4] = 0$	$(1+\alpha^2)\beta'' - 2\alpha\beta' + 2\beta = -1$
$[V_2, V_5] = 0$	$\alpha^2(1+\alpha^2)\beta'' + 2\alpha(2+\alpha)\beta' + 2\beta = 0$
$[V_3, V_4] = 0$	$\beta\beta'' - 2\beta'^2 - \beta^3 = 0$
$[V_3, V_5] = 0$	$\beta'' = 0$

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